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STRENGTH OF A JOINT UNDER STRESS CONCENTRATION

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In structural design, investigation of the low stress problem for the end of a contact surface in a composite body allows one to ensure the reliable strength of the given joint by a proper choice of physical and geometrical parameters [1, 2]. However, at fixed values of the indicated parameters, the low stress condition may not be satisfied, and a concentrated stressed state occurs at this end. Thus, it becomes necessary to formulate strength conditions in the presence of stress concentration.

In this work, using the sectioning method, which is well known in engineering mechanics, we examine the strength of a joint in composite bodies with exponential strengthening of materials in the presence of stress concentration. The use of the sectioning method in the linear mechanics of cracks is covered in [3].

Let a composite body be made of two dissimilar materials. The stress and strain intensities are related by the exponential dependence $\sigma_0 = k\varepsilon_0^m$. For both materials, the values of the parameter m are considered equal and the values of k different.

We assume that at the end of the contact surface of the composite body there is a reentrant angular "cut" with stress concentration at the tip. Assume that we know a solution of the corresponding problem ignoring the stress concentration caused by the "cut." This stressed state will be called nominal. Further, using this solution for the neighborhood of the angular point, we should find the strength condition for the end of the joint with the stress concentration state.

1. Twisting. We consider a composite bar of constant cross section made of materials strengthening by an exponential law. The bar has a reentrant angle at the end of the contact surface and is twisted by the moments M applied at the end cross sections.

Initial Relations. We assume that the nominal solution, i.e., without an angular cut, is specified in the polar coordinate system $\rho\varphi$ (Fig. 1). We denote stresses by $T_{\rho i}(\rho,\varphi)$ and $T_{\varphi i}(\rho,\varphi)$, and displacements by $W_i(\rho,\varphi)$. Here and below, the subscript $i = 1$ and 2 denotes the values of the constituent materials.

Longitudinal shear strains occur in the vicinity of the angular point $r = 0$.

According to [2], this solution is representable as

$$\tau_{\theta i} = k_i r^{(\lambda-1)m} f_i' \chi_i, \quad \tau_{ri} = \lambda k_i r^{(\lambda-1)m} f_i \chi_i, \quad w_i = r^\lambda f_i, \quad \chi_i = \left(\sqrt{f_i'^2 + \lambda^2 f_i^2}\right)^{m-1}, \tag{1.1}$$

in which

$$f_1 = A \exp\left(-\int_0^\alpha \psi_1 d\theta\right); \qquad f_2 = A \exp\left(-\int_0^\alpha \psi_1 d\theta - \int_\theta^0 \psi_2 d\theta\right),$$

$A = f_1(\alpha)$ is an arbitrary constant, and $\psi_1(\theta,\lambda)$ is found from the relations

$$\arctan\frac{\psi_1}{\lambda} + \frac{1-\lambda}{\omega}\arctan\frac{\psi_1}{\omega} = \alpha - \theta \quad \text{for} \quad 0 \leqslant \theta \leqslant \alpha,$$

$$\arctan\frac{\psi_2}{\lambda} + \frac{1-\lambda}{\omega}\arctan\frac{\psi_2}{\omega} = \pi\left(1 + \frac{1-\lambda}{\omega}\right) - \beta - \theta \quad \text{for} \quad \frac{\pi}{2}\left(1 + \frac{1-\lambda}{\omega}\right) - \beta \leqslant \theta \leqslant 0,$$

where $\omega = \sqrt{\lambda(\lambda + n - 1)}$ and $n = 1/m$

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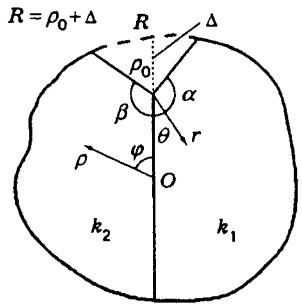

Fig. 1

The parameter λ is determined from the following system of three equations in $\mu_i = \psi_i(0,\lambda)$ and λ:

$$\mu_1\left(\sqrt{\mu_1^2+\lambda^2}\right)^{m-1} - \gamma\mu_2\left(\sqrt{\mu_2^2+\lambda^2}\right)^{m-1} = 0, \quad \arctan\frac{\mu_1}{\lambda} + \frac{1-\lambda}{\omega}\arctan\frac{\mu_1}{\omega} = \alpha,$$
$$\arctan\frac{\mu_2}{\lambda} + \frac{1-\lambda}{\omega}\arctan\frac{\mu_2}{\omega} = \pi\left(1+\frac{1-\lambda}{\omega}\right) - \beta \quad (\gamma = k_2/k_1). \tag{1.2}$$

If we introduce the notation $s_1 = 2\alpha/\pi - 1$ and $s_2 = 2\beta/\pi - 1$, the solution of system (1.2) takes the form $\lambda = \lambda(s_1, s_2, \gamma, m)$. When the angle is symmetric about the contact surface, i.e., $\alpha = \beta$, or when the material is homogeneous, λ is defined as

$$\lambda = \frac{2 + s\left[(n-1)s - \sqrt{4n + (n-1)^2 s^2}\right]}{2(1-s^2)} \quad \text{for} \quad s \neq 1,$$
$$\lambda = 1/(n+1) \quad \text{for} \quad s = 1. \tag{1.3}$$

Here $s = 2\alpha/\pi - 1$.

The second value of λ in (1.3) also follows from the first formula in the limit $s \to 1$.

We assume that, at the tested end of the contact surface, the conditions $\lambda < 1$ and $\pi < \alpha + \beta \leqslant 2\pi$ or $0 < s_1 + s_2 \leqslant 2$ are satisfied. For the symmetric angle, we have $\pi/2 < \alpha \leqslant \pi$ or $0 < s \leqslant 1$.

Fracture Surface. Introducing the notation

$$N = k_i f_i'(0)\chi_i(0), \qquad F_i(\theta) = f_i'(\theta)\chi_i(\theta)/(f_i'(0)\chi_i(0)),$$

we write the stress components (1.1) in the form

$$\tau_{\theta i} = N r^{(\lambda-1)m} F_i(\theta), \qquad \tau_{ri} = \lambda f_i \tau_{\theta i}/f_i'. \tag{1.4}$$

The desired constant $N = \lim\limits_{r\to 0} \tau_{\theta i}(r,0)\, r^{(1-\lambda)m}$ has dimension $\text{kg}\cdot\text{cm}^{(1-\lambda)m-2}$ and is similar in some sense to the stress-intensity coefficient at the crack tip. The coefficient N can be determined approximately using the above-mentioned sectioning method.

We pass mentally a section through the plane $\varphi = 0$, and, rejecting one part of the bar, we analyze the equilibrium of the remaining part under the action of longitudinal tangential forces acting in the axial section for a bar with no reentrant angle and for a bar with a reentrant angle. We assume that, owing to the angular cut, the longitudinal force decreases by a value equal to the contribution of the concentrated forces.

Projecting the tangential forces in the longitudinal direction and equating the sum of the minimal forces acting in the interval $\rho_0 \leqslant \rho \leqslant R$ in a bar with no angular cut to the sum of the concentrated forces acting in the interval $0 \leqslant r \leqslant r_0$ in a bar with an angular cut, we find

$$\int_0^{r_0} \tau_{\theta i}(r,0)\,dr = \int_{\rho_0}^{R} T_{\varphi i}(\rho,0)\,d\rho. \tag{1.5}$$

Here r_0 is the unknown distance from the angular point, at which the concentrated stress is equated to the nominal stress:

$$\tau_{\theta i}(r_0,0)=T_{\varphi i}(\rho_0,0). \tag{1.6}$$

In Eqs. (1.5) and (1.6), according to [3], the nominal stress is taken at the angular point and the nominal forces in the interval of concentrated stress are ignored. These assumptions somehow compensate for the decreased asymptotic value of the local stress compared with its exact value.

Substituting the expression of $\tau_{\theta i}(r,0)$ from (1.4) into (1.5) and (1.6), we have

$$N=\tau r_0^{(1-\lambda)m}, \qquad r_0=[1+(\lambda-1)m]\int\limits_{\rho_0}^{R}\frac{T_{\varphi i}(\rho,0)}{T_{\varphi i}(\rho_0,0)}\,d\rho, \tag{1.7}$$

where $\tau=T_{\varphi i}(\rho_0,0)$ is the nominal stress at the angular point considered. Introducing the notation $\rho=\xi R$ and $R-\rho_0=\Delta=\delta R$, from (1.7), we obtain

$$N=\tau\{\Delta[1+(\lambda-1)\,m]\Phi(\delta,\gamma,m)\}^{(1-\lambda)m}. \tag{1.8}$$

Here

$$\Phi=\frac{1}{\delta}\int\limits_{1-\delta}^{1}H(\xi;\delta,\gamma,m)\,d\xi. \tag{1.9}$$

The function $H=H_i$ does not depend on the external moment M and is determined by the integrand (1.7) with the above replacements of the dimensionless variable and parameters.

The critical value of τ for which the joint failed at the angular point considered is denoted by τ_*, and the corresponding value of M is denoted by M_*. The parameter τ_*, which depends on the constituent materials, the realization of the joint, the binding material, the manufacture of surfaces for the contact, etc., is determined from experiments. For a homogeneous material, τ_* is determined by the fracturing twisting moment M_*.

According to (1.8), the critical value of N is written as

$$N_*=\tau_*\{\Delta[1+(\lambda-1)\,m]\Phi\}^{(1-\lambda)m}. \tag{1.10}$$

In the space of our parameter, the resulting equation (1.10), i.e., $N_*=N_*(s_1,s_2,\Delta,\delta,m)$, can be interpreted as a fracture hypersurface. For fixed values of the parameters Δ, δ, γ, and m, the equation $N_*=N_*(s_1,s_2)$ defines the limiting fracture surface in the three-dimensional space $s_1s_2N_1$. This surface separates the strength region (below the surface) from the fracture region (above the surface). When the angle is symmetric, in the coordinate plane sN_*, we have the limiting fracture curve $N_*=N_*(s)$, which separates the strength and fracture regions.

Using (1.4) and (1.10), we can write the critical value of the contact stress as

$$\tau^*_{\theta i}(r,0)=\tau_*\left(\frac{\Delta}{r}\right)^{(1-\lambda)m}\{[1+(\lambda-1)\,m]\Phi\}^{(1-\lambda)m}. \tag{1.11}$$

Twisting of a Cylindrical Tube. Let us study the twisting of a long, thick-walled cylindrical tube of an exponentially strengthening, homogeneous material that has an angular cut on the outer surface (Fig. 2). The nominal stress is known [4]:

$$T_\varphi(\rho)=\frac{(m+3)\,Mh}{2\pi R^{m+3}}\,\rho^m, \quad h=\left[1-\left(\frac{a}{R}\right)^{m+3}\right]^{-1}$$

Here R and a are the radii of the outer and inner surfaces of the tube. We also have

$$\tau=T_\varphi(\rho_0)=\frac{(m+3)\,Mh}{2\pi R^{m+3}}\,\rho_0^m, \qquad H=\rho^m/\rho_0^m.$$

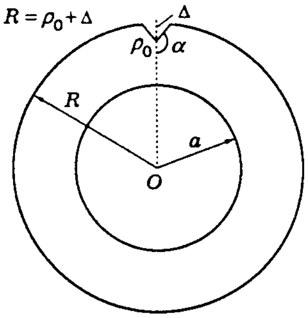

Fig. 2

Integrating (1.9) yields

$$\Phi = \frac{1-(1-\delta)^{m+1}}{(m+1)\delta(1-\delta)^m}. \tag{1.12}$$

It is evident that $\Phi \to 1$ as $\delta \to 0$.

The critical value of N in (1.10) is written as

$$N_* = \tau_* \left\{ \frac{\Delta[1+(\lambda-1)m][1-(1-\delta)^{m+1}]}{(m+1)\delta(1-\delta)^m} \right\}^{(1-\lambda)m}. \tag{1.13}$$

Here

$$\tau_* = \frac{(m+3)\,M_* h}{2\pi R^3}\,(1-\delta)^m, \tag{1.14}$$

and λ is defined by (1.3).

For fixed values of Δ, δ, and m, Eq. (1.13) defines the limiting fracture curve $N_* = N_*(s)$, which separates the strength and fracture regions.

In the case of a slit (crack), i.e., for $s = 1$, setting $\lambda = m/(m+1)$ in (1.13), we find

$$N_* = \tau_* \left\{ \frac{\Delta[1-(1-\delta)^{m+1}]}{(m+1)^2\delta(1-\delta)^m} \right\}^{m/(m+1)}. \tag{1.15}$$

For values of δ that are small compared with unity, it follows from (1.13) and (1.14) that

$$N_* = \frac{(m+3)\,M_* h}{2\pi R^3}\,\{\Delta[1+(\lambda-1)\,m]\}^{(1-\lambda)m}.$$

Hence, for the case of a slit, we obtain

$$N_* = \frac{(m+3)\,M_* h}{2\pi R^3}\left(\frac{\Delta}{m+1}\right)^{m/(m+1)}. \tag{1.16}$$

When the material is linearly elastic, assuming that $m = 1$ and $\lambda = 1/(s+1)$ in (1.13), we have

$$N_* = \frac{2(1-\delta)\,M_*}{\pi R^3[1-(a/R)^4]}\left[\frac{\Delta(1-\delta/2)}{(1-\delta)(s+1)}\right]^{s/(s+1)}. \tag{1.17}$$

In the case of a slit, assuming that $s = 1$, from (1.17) we find

$$N_* = \frac{\sqrt{2}M_*}{\pi R^{5/2}}\,\frac{\sqrt{\delta}\sqrt{(1-\delta)(1-\delta/2)}}{1-(a/R)^4}. \tag{1.18}$$

Hence, for small δ or from (1.16) for $m = 1$, we obtain

$$N_* = \frac{\sqrt{2}}{\pi}\,\frac{M_* R^{-5/2}\sqrt{\delta}}{1-(a/R)^4}. \tag{1.19}$$

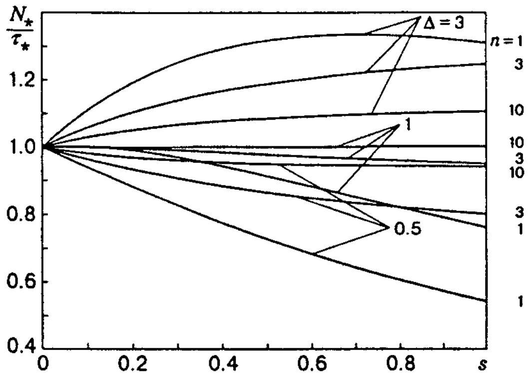

Fig. 3

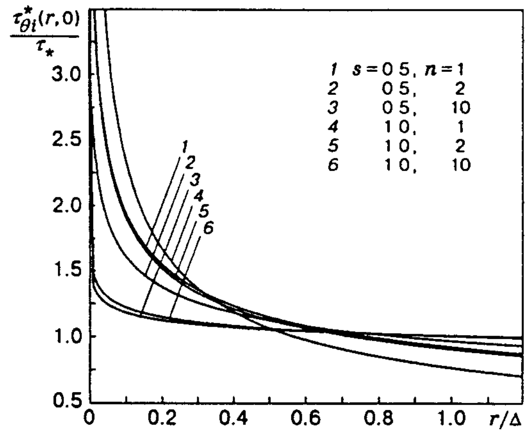

Fig. 4

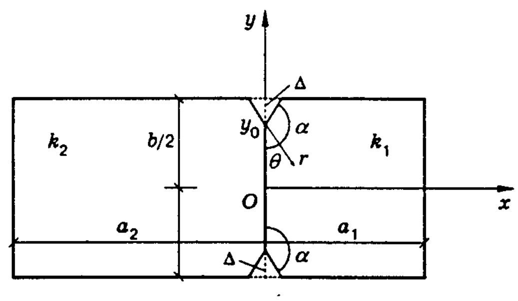

Fig. 5

Assuming that $a/R = 0.5$ and calculating M_* for $\delta = 0.05$ by formula (1.19) and for $\delta = 0.2$ by (1.18), we find $N_* = 0.1M_*R^{-5/2}$ and $N_* = 0.18M_*R^{-5/2}$, respectively. Comparison of these formulas (after multiplication by $\sqrt{2\pi}$) with the exact solutions given in the form of plots in [5, p. 723] shows that the results obtained herein are underestimated by about 10%.

Figure 3 shows the limiting curves constructed by formula (1.13) for $\delta = 0.2$ at fixed values of the parameters Δ and m.

The critical contact stresses obtained from (1.11) and (1.12) for $\delta = 0.2$ are given in Fig. 4.

Composite Linear-Elastic Bar. Let a bar to be twisted consist of two linear-elastic rectangular bars connected by complete slipping along the lateral surfaces. At the ends of the contact surface there are two identical reentrant angles that are symmetric about this surface (Fig. 5).

This problem without angular cuts is generally solved in [6]. According to [6], the nominal contact pressure is written as

$$T_{xi}(0,y) = \frac{M}{Db^3}\left[4\sum_{n=0}^{\infty}\frac{(-1)^n\Omega_n(\gamma)}{p_n^2}\sin p_n\frac{y}{b} - \frac{y}{b}\right],$$

where $\Omega_n = (1/\omega_n)\{[\gamma+(1-\gamma)\cosh p_n c_2]\sinh p_n c_1+\gamma\sinh p_n c_2\}$, $\omega_n = \cosh p_n c_2 \sinh p_n c_1 + \gamma\cosh p_n c_1 \sinh p_n c_2$, $p_n = (2n+1)\pi/2$, $\gamma = k_2/k_1$, $c_i = a_i/b$, k_i are the shear moduli of the materials, $D = (8/3)(c_1 + \gamma c_2) + 32\sum_{n=0}^{\infty}\frac{Q_n - \gamma R_n}{p_n^5\omega_n}$, $Q_n = \cosh p_n c_2 + \gamma^2\cosh p_n c_1 - (1+\gamma^2)\cosh p_n c_1 \cosh p_n c_2$, and $R_n = \cosh p_n c_1 + \cosh p_n c_2 -$

$\cosh p_n(c_1 + c_2) - 1$. The nominal pressure at the angular point is $\tau = T_i(0, y_0) = MS(\gamma, \delta)/(Db^3)$. Here

$$S = 4 \sum_{n=0}^{\infty} \frac{(-1)^n \Omega_n(\gamma)}{p_n^2} \sin p_n(1 - \delta) + \delta - 1; \quad b - y_0 = \Delta = \delta b.$$

The critical value of τ is expressed in terms of the fracturing moment as $\tau_* = M_* S/Db^3$.

Next, calculating integral (1.9), introducing the variable $\xi = y/b$, and assuming that $\lambda = 1/(s + 1)$, from (1.8) we obtain

$$N_* = \tau_* \left[\frac{\Delta}{s+1} \Phi(\gamma, \delta)\right]^{s/(s+1)}, \tag{1.20}$$

where

$$\Phi = \frac{1}{S} \left[\frac{8}{\delta} \sum_{n=0}^{\infty} \frac{(-1)^n \Omega_n(\gamma)}{p_n^3} \sin \frac{p_n}{2} (2 - \delta) \sin \frac{p_n}{2} \delta + \frac{1}{2} \delta - 1\right].$$

Note also that $\Phi \to 1$ as $\delta \to 0$. For $s = 1$, i.e., for a slit, relation (1.20) leads to the relation $N_* = \tau_* \sqrt{\Delta/2} \sqrt{\Phi(\gamma, \delta)}$. For fixed value of the parameters Δ, δ, and γ, Eq. (1.20) gives the limiting fracture curve, which separates the strength and fracture regions.

2. Plane Deformation. Let a composite body with exponential strengthening of the materials be in the state of plane deformation. We assume that at the end of the contact surface there is a reentrant angle that is free of external forces and stress concentrations. For the case of plane deformation, we shall also use Fig. 1.

Initial Relations and Equations. Let the solution of the general problem, i.e., without a reentrant angle, be known. The stress components $T_{\rho i}$, $T_{\varphi i}$, and $T_{\rho\varphi i}$ and the displacement components u_i and v_i, which are still called nominal, are assumed to be given in the polar coordinate system $\rho\varphi$. On the other hand, we shall proceed from the local solution at the angular point considered in the polar coordinate system $r\theta$ given in [2]. In this solution, we shall replace the functions $f_i(\theta, \lambda)$ by $Af_i(\theta, \lambda)$, where A is an arbitrary constant and $f_i(\theta, \lambda)$ is a solution of the following system of differential equations:

$$[(f_i'' + \mu f_i)\chi_i]'' + \nu(f_i'' + \mu f_i)\chi_i + 4\eta(f_i'\chi_i)' = 0. \tag{2.1}$$

Here $\chi_i = \left(\sqrt{(f_i'' + \mu f_i)^2 + 4\lambda^2 f_i'^2}\right)^{(m-1)}$, λ is the desired parameter, $\eta = \lambda[1 + (\lambda - 1)m]$, $\mu = 1 - \lambda^2$, and $\nu = 1 - \eta^2/\lambda^2$. If we introduce the notation $\sigma_i(\theta) = (1/(\lambda - 1)m)\left\{[(f_i'' + \mu f_i)\chi_i]' + 4\eta f_i'\chi_i\right\}$ and $\tau_i(\theta) = (f_i'' + \mu f_i)\chi_i$, on the edges that are free from external forces, the boundary-contact conditions are of the following form:

on the outer edges,

$$\sigma_i = \tau_i = 0 \quad \text{for} \quad \theta = \alpha; \ -\beta; \tag{2.2}$$

and on the contact surface,

$$\sigma_1 = \gamma\sigma_2, \quad \tau_1 = \gamma\tau_2, \quad f_1' = f_2', \quad f_1 = f_2 = 1 \quad \text{for} \quad \theta = 0. \tag{2.3}$$

In this case, the normalization condition is adopted without loss of generality. As before, $\gamma = k_2/k_1$, where k_i are strain moduli of the materials. We assume that the problem of finding the function $f_i(\theta, \lambda)$ and the parameter λ from (2.1)–(2.3) is solved. Using the previous notation, we assume that the value of $\lambda = \lambda(s_1, s_2, \gamma, m)$ is determined by a numerical or other method. In the case of a slit and a homogeneous material [7, 8], we have $\lambda = m/(m + 1)$. In the general case, we assume that the angle is reentrant, and $\lambda < 1$, i.e., we have stress concentrations.

Fracture Surface. Introducing the notation $A|A|^{m-1}k_i\Omega_i = N$, $\Omega_i = \sqrt{\sigma_i^2(0) + \tau_i^2(0)}$, we write the contact stresses [2] in the vicinity of the angular point as

$$\sigma_{\theta i}(r, 0) = N r^{(\lambda-1)m} \sigma_i(0)/\Omega_i, \qquad \tau_{r\theta i}(r, 0) = N r^{(\lambda-1)m} \tau_i(0)/\Omega_i. \tag{2.4}$$

The constant

$$N = \frac{\Omega_i}{\sigma_i(0)} \lim_{r \to 0} \sigma_{\theta i}(r,0)\, r^{(1-\lambda)m} = \frac{\Omega_i}{\tau_i(0)} \lim_{r \to 0} \tau_{r\theta i}(r,0) r^{(1-\lambda)m},$$

which has dimension $\mathrm{kg} \cdot \mathrm{cm}^{(1-\lambda)m-2}$ and is to be determined, is similar to the stress-intensity coefficient in the expressions of stresses at the crack tip.

Next, we introduce the intensity of nominal stresses on the contact surface $T(\rho) = \sqrt{T^2_{\varphi i}(\rho,0) + T^2_{\rho\varphi i}(\rho,0)}$, and, similarly, the intensity of concentrated stresses, $p(r) = \sqrt{\sigma^2_{\theta i}(r,0) + \tau^2_{r\theta i}(r,0)} = N r^{(\lambda-1)m}$.

At an unknown distance r_0 from the angular point on the contact surface the concentrated-stress intensity are equated to the nominal-stress intensity $p(r_0) = T(\rho_0)$, which leads to the equation

$$N r_0^{(\lambda-1)m} = T(\rho_0), \tag{2.5}$$

which contains the unknown constant N and r_0.

To derive the second equation, according to the sectioning method, we mentally pass a section through the plane $\varphi = 0$, and, rejecting one part of the body, we analyze the equilibrium of the remaining part. We assume that, owing to the angular cut, which does not transfer stresses in the interval $\rho_0 \leqslant \rho \leqslant R$, the force is transferred by stresses with a singularity at the angular point. For an intact body without an angle, the normal and tangential nominal stresses acting in the indicated interval must be equilibrated by the concentrated and tangential stresses acting on the contact surface in the interval $0 \leqslant r \leqslant r_0$ in the vicinity of the angular point.

In the particular case where one of the two stress components is absent on the contact surface, we have one equilibrium equation, which is sufficient, together with (2.5), to determine N and r_0.

To find the second equation in the general case, we consider the function

$$U = P^2 + Q^2, \tag{2.6}$$

where

$$P = \int\limits_{\rho_0}^{R} T_{\varphi i}(\rho,0)\, d\rho - \int\limits_{0}^{r_0} \sigma_{\theta i}(r,0)\, dr, \quad Q = \int\limits_{\rho_0}^{R} T_{\rho\varphi i}(\rho,0)\, d\rho - \int\limits_{0}^{r_0} \tau_{r\theta i}(r,0)\, dr. \tag{2.7}$$

Here P is the difference of the normal nominal and concentrated forces acting in the above-mentioned intervals of the contact surface and Q is the difference of the tangential nominal and concentrated forces acting in the same intervals.

Substituting expressions $\sigma_{\theta i}(r,0)$ and $\tau_{r\theta i}(r,0)$ from (2.4) into (2.7), integrating, and then eliminating r_0, according to (2.5), we find

$$P = \int\limits_{\rho_0}^{R} T_{\varphi i}(\rho,0)\, d\rho - N_0 B \frac{\sigma_i(0)}{\Omega_i}, \quad Q = \int\limits_{\rho_0}^{R} T_{\rho\varphi i}(\rho,0)\, d\rho - N_0 B \frac{\tau_i(0)}{\Omega_i}, \tag{2.8}$$

where

$$N_0 = N^{1/(1-\lambda)m}; \quad B = \bigl(1/[1 + (\lambda - 1)\, m]\bigr)\, [T(\rho_0)]^{1+1/(1-\lambda)m}.$$

Substituting (2.8) into (2.6), defining the first derivative of U with respect to N_0 as

$$\frac{\partial U}{\partial N_0} = -2B \left[P \frac{\sigma_i(0)}{\Omega_i} + Q \frac{\tau_i(0)}{\Omega_i} \right], \tag{2.9}$$

and equating to zero, we find N_0 and then N in the form

$$N = T(\rho_0) \Bigl\{ [1 + (\lambda - 1)\, m] \int\limits_{\rho_0}^{R} \frac{\sigma_i(0) T_{\varphi i}(\rho,0) + \tau_i(0) T_{\rho\varphi i}(\rho,0)}{\Omega_i T(\rho_0)}\, d\rho \Bigr\}^{(1-\lambda)m}. \tag{2.10}$$

It is easy to see that $\partial^2 U/\partial N_0^2 = 2B^2 > 0$. This means that the N determined from (2.10) provides a minimal value of the function U.

The critical value of N is written as

$$N_* = T_*\{\Delta[1 + (\lambda - 1)m]\Phi(\delta, \gamma, m)\}^{(1-\lambda)m}, \tag{2.11}$$

where

$$\Phi = \frac{1}{\delta} \int\limits_{1-\delta}^{1} H(\xi; \delta, \gamma, m)\, d\xi. \tag{2.12}$$

Here the function H is given by the integrand (2.10) with passage to the dimensionless variable and parameters, and T_* denotes the parameter $T(\rho_0)$ for which fracture of the joint occurs at the angular point. The parameter T_* is determined experimentally by measuring the fracturing external forces at which fracture of the material occurs at the angular point.

The critical values of stresses on the contact surface with allowance for (2.11) can be written as

$$\frac{\sigma^*_{\theta i}(r,0)}{\sigma_i(0)} = \frac{\tau^*_{r\theta i}(r,0)}{\tau_i(0)} = \frac{T_*}{\Omega_i}\left(\frac{\Delta}{r}\right)^{(1-\lambda)m}\{[1 + (\lambda - 1)m]\Phi(\delta, \gamma, m)\}^{(1-\lambda)m}.$$

Cylindrical Tube under Internal Pressure. We consider a long, thick-walled homogeneous tube that is made of an exponentially strengthening material and has a radial symmetric angular cut on the inner surface. Let the tube be under uniform internal pressure p (in Fig. 2, $\sigma_r = -p$ is added).

The nominal pressure, according to [9], is written as

$$T_\varphi(\rho) = \frac{pl}{1-l}\left[1 + (2m-1)\left(\frac{R}{\rho}\right)^{-2m}\right], \qquad l = \left(\frac{a}{R}\right)^{2m}.$$

Here a and R are the radii of the outer and inner surface of the tube and $R - \rho_0 = \Delta = \delta R$. The nominal pressure at the angular point is $T_\varphi(\rho_0) = [pl/(1-l)][1 + (2m-1)(1-\delta)^{-2m}]$.

Calculating the integral from (2.10) and (2.12) and taking into account that $\tau_i(0) = 0$, we find

$$\Phi = \frac{(1-\delta)[1-(1-\delta)^{2m}]}{\delta[2m - 1 + (1-\delta)^{2m}]}. \tag{2.13}$$

Hence it follows that $\Phi \to 1$ as $\delta \to 0$. From (2.11) we have

$$N_* = T_*\{\Delta[1 + (\lambda - 1)m]\Phi(\delta, m)\}^{(1-\lambda)m}, \tag{2.14}$$

where $\lambda = \lambda(s, m)$ and $T_* = [p_* l/(1-l)][1 + (2m-1)(1-\delta)^{-2m}]$. Here p_* is the critical pressure at which fracture at the angular point considered occurs; T_* is determined experimentally (by measuring p_*).

For fixed parameters δ and m, Eq. (2.14), i.e., $N_* = N_*(s, \delta, m)$, gives the limiting fracture curve, which separates the strength and fracture regions. For the case of a slit, assuming that $\lambda = m/(m+1)$ [7, 8], we obtain

$$N_* = T_*\left[\frac{\Delta}{m+1}\Phi(\delta, m)\right]^{m/(m+1)}. \tag{2.15}$$

Taking into account the value of the function Φ (2.13) for small values of δ, from (2.15) we have

$$N_* = q_*\left(\frac{\Delta}{m+1}\right)^{m/(m+1)}, \qquad q_* = \frac{2mp_*(a/R)^{2m}}{1-(a/R)^{2m}}. \tag{2.16}$$

Figure 6 shows plots of N_*/q_* versus m for three values of the parameter Δ. In the case of a crack in a linear-elastic material, from (2.13) and (2.14) we find

$$N_* = \frac{\sqrt{2}p_*(a/R)^2\sqrt{\Delta}}{1-(a/R)^2}\,\frac{\sqrt{(1-\delta/2)(1-\delta+\delta^2/2)}}{(1-\delta)^{3/2}}.$$

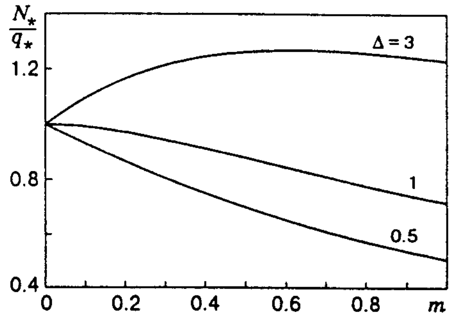

Fig. 6

Hence, for small values of δ, or from (2.16) for $m = 1$ we have

$$N_* = \frac{\sqrt{2}p_*(a/R)^2\sqrt{\Delta}}{1-(a/R)^2}.$$

Comparison of the resulting formula for small values of δ (after multiplication by $\sqrt{2\pi}$) with the corresponding exact value [10, p. 320] (for a circular ring) shows a decrease by about 10%. With increase in δ, this difference increases.

The indicated deviations of the values of N from the exact values are of a one-sided character — toward underestimation (see also [3]). Therefore, the approximate values of N obtained herein can be considered as lower bounds.

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